Separable Social Welfare Evaluation for Multi-Species Populations*

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Abstract

If non-human animals experience wellbeing and suffering, such welfare consequences arguably should be included in a social welfare evaluation. Yet economic evaluations almost universally ignore non-human animals, in part because axiomatic social choice theory has failed to propose and characterize multi-species social welfare functions. Here we propose axioms and functional forms to fill this gap. We provide a range of alternative representations, characterizing a broad range of possibilities for multi-species social welfare. Among these, we identify a new characterization of additively-separable generalized (multi-species) total utilitarianism. The multi-species setting permits a novel, weak species-level separability axiom with important consequences. We provide examples to illustrate that non-separability across species is implausible in a multi-species setting, in part because good lives for different species are at very different welfare levels. Finally, we explore the consequences for evaluating climate policy and understanding speciesism and non-climate environmental goals, such as biodiversity.

Keywords: animals, animal welfare, social welfare functions, population ethics, utilitarianism

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1 Introduction

If non-human animals experience wellbeing and suffering, such welfare consequences arguably should be included in a social welfare evaluation. Welfarists from Bentham (1780) to Singer (1975) have recognized that welfarist social evaluation must incorporate animal welfare. Yet exactly how to do so is a deeply open question—even compared with the many open question in human welfare economics (Blackorby, Bossert and Donaldson, 2005). Economic evaluations almost universally ignore non-human animals, in part because axiomatic social choice theory has failed to propose and characterize multi-species social welfare functions. Which multi-species social welfare functions are consistent with attractive normative principles? Finding an answer is important for many economic policy questions including climate change, food policy, and medical research.

We contribute the first axiomatic characterization of a multi-species social welfare function, using explicitly multi-species axioms. Some prior studies, including Blackorby and Donaldson (1992), the pioneer welfare economics research to incorporate non-human animals, as well as Clarke and Ng (2006), Budolfson and Spears (2019), and Espinosa and Treich (2021), articulate a multi-species social welfare function without characterizing it and then use it to address economic or ethical questions. Others, including Cowen (2006) and Fleurbaey and Van der Linden (2021), build upon the observation that animal welfare has implications for welfarist social evaluation if some humans care about animals.

We characterize social orderings of variable-population, multi-species vectors of individual lifetime utilities. We focus particularly on the arguments for, possibilities for, and alternatives to cross-species separability (even while, in some characterizations, permitting within-species non-separability). It is well-known in welfare economics that assuming individual-level independence yields a fully additively-separable, generalized utilitarian social welfare function. We show that weaker, species-level separability achieves the same utilitarian outcome, in the context of individual-level anonymity. This result adds to the debate about separability in the welfare economics literature and increases the theoretical cost of choosing a non-separable form. The possibility of welfare-relevant non-human animals, this result suggests, offers a reason for social evaluation to be additively-separable across individuals, even when comparing a policy choice that would only have consequences for humans. Moving away from separability or anonymity, we show, permits other functional forms.

In the next section we offer simple numerical examples. We intend these to motivate the intuition for cross-species separability. The next, main sections of the paper present our characterization results. In addition to separability and anonymity, we explore a complementarity axiom which proposes that, in allocating two species population sizes between two species, each with a different perfectly-egalitarian per-person welfare level, it is better to assign the larger population size to the species with better-off individuals. We conclude with a discusion of possible implications, including for climate policy and measures of biodiversity.

2 Motivating examples

Here we propose that species-level independence is attractive in a multi-species, variable-population setting, in part because different species have very different utilities for individuals living a good life. As a result, there is no clear "denominator" that a non-separable evaluation could use to combine individuals of very different species.

First consider generalized, multi-species average utilitarianism of the form: $\frac{\sum_s \Xi_n^s \times n(s)}{\sum_s n(s)}$, where s indexes species, Ξ_n^s is the equally-distributed-equivalent utility of species s of size n, and $n(s)$ is the size of species s. Average utilitarianism is separable across species for fixed-population cases but non-separable for variable population cases because of its denominator. Assume the world consists of:

- 1 million mammals, each living great lives for a mammal at utility 10;
- 400,000 birds, each living great lives for a bird at utility 5; and
- 300,000 lizards, each living great lives for a lizard at utility 3.

Would it be an improvement or a worsening to add 10 birds each at utility 6? Assume that this utility level, although lower than that of the mammals (perhaps because it represents a shorter life), is an excellent life for a bird. According to average utilitarianism, adding these birds, which lower average welfare, would be a worsening — so it would be better to slightly worsen the lives of each mammal and lizard in order to prevent these birds from being born. We find this implausible.

Averagism is not unique in these implications. Consider the following example of species-blind, rank-dependent (and therefore non-separable) utilitarianism: $\sum_r \beta^r u_r$, where r is an individual's wellbeing rank from the worst-off, u_r is the individual's lifetime utility, and β is a constant between 0 and 1. For simplicity, imagine a world containing:

- 1 mammal, living a great mammal life at 10;
- 1 bird, living a great bird life at 5; and

- 1 lizard, living a great lizard life at 3.

Would it be an improvement or a worsening to improve the bird's life to 5.1, add a second bird also at 5.1, and leave the mammal and the lizard unchanged? If $\beta = \frac{1}{3}$, then such an improvement-and-addition would make the population worse — again recommending harming the mammal and the lizard, if necessary, to prevent it. In this case, the same recommendation — rejecting the improvement-and-addition — could be made by species-blind, number-sensitive egalitarianism of the form $n^{\alpha} \times g^{-1}\left(\frac{1}{n}\sum g(u_i)\right)$ for a concave g such as $g(u) = \ln(u)$, even with some positive values for the constant α.

Should there be no happy dogs in a world of happy humans, merely because the dogs' lives are shorter? Should there be no humans in a galaxy of blissful aliens? Perhaps not, according to average utilitarianism, rank-dependent utilitarianism, and egalitarianism.

A multi-species setting offers a new version of classic arguments for independence, such as by Blackorby, Bossert and Donaldson (2005). What these examples reveal is a consequence of the fact that different species can have very different welfare profiles, even in good lives. A life can be good *for a species* without being good *relative to the full, multi-species population*. But it strikes us as normatively implausible — and arguably speciesist — to consider an extra individual a worsening when its own life is worth living, is excellent by the distribution of its species, and has a highly favorable balance of pleasure and suffering. This speciesist outcome is not escaped merely by a social welfare function being species-blind. The fact that these and other implausible implications can be avoided by assuming independence across species motivates our novel axioms.

3 Framework

The set of positive integers is denoted by $\mathbb{N}$. The set of all real numbers is denoted by $\mathbb{R}$. The set of non-negative (resp. positive) real numbers is denoted by $\mathbb{R}_+$ ($\mathbb{R}_{++}$).

There is a finite set of species S. For simplicity, we assume that $S = \{1, \cdots, T\}$ for some $T \in \mathbb{N}$ with $T \geq 3$. For each species, a variable number of individuals may exist. If population size is $n \in \mathbb{N}$, the wellbeing vector for individuals is $u = (u_1, \cdots, u_n)$, where u_i is individual i's the lifetime utility. The set of all possible wellbeing vector is $U = \bigcup_{k\in\mathbb{N}} \mathbb{R}^k$. For $u \in U$, we write $n(u)$ the population size, that is $n(u) = k$ if $u \in \mathbb{R}^k$, and $N(u) = \{1, \cdots, n(u)\}$. For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, we denote $x \cdot \mathbb{1}_k = (x, \cdots, x) \in \mathbb{R}^k$.

An alternative will give a wellbeing vector for each species. This can be modeled as a mapping $a : S \to U$, where for each $s \in S$ $a(s) \in U$ is the wellbeing vector in species s. We denote A the set of all possible mappings $a : S \to U$, so that A is the set of alternatives.

When $a(s) = u \in U$, we denote $a_i(s) = u_i$ for each $i \in N(a(s))$ the wellbeing of individual i of species s in alternative a.

For a subset of species $\bar{S} \subset S$ and for two alternatives $a, a' \in A$, we denote $a_{\bar{S}}a'$ the alternative $\hat{a} \in A$ such that $\hat{a}(s) = a(s)$ for all $s \in \bar{S}$ and $\hat{a}(s) = a'(s)$ for all $s \in (S \setminus \bar{S})$. When $\bar{S} = \{s\}$ we write $a_s a'$ instead of $a_{\{s\}}a'$.

For any $a \in A$, let us denote $\mathbf{n}(a) = \big(n(a(1)), \cdots, n(a(T))\big)$ the vector of species population sizes. For $\mathbf{n} = (n_1, \cdots, n_T)$ an element of $\mathbb{N}^T$, we denote $A_{\mathbf{n}}$ the set of alternatives in A such that $\mathbf{n}(a) = \mathbf{n}$. These are alternatives with a given population size n_s for each species s.

3.1 Basic principles

We study a social ordering $\succsim$ on A. We assume two basic principles throughout. The first is a Pareto principle applied to situations where population size is fixed for each species. It says that if all individuals of each species is at least as well off and all individual of some species are better off, then the situation is socially better. Hence we do not necessarily judge that an improvement for one individual is enough to increase social welfare, but an improvement for all individuals in a species is enough.

Pareto. For all $a, a' \in A$, if $n(a(s)) = n(a'(s))$, $a(s) \geq a'(s)$ for all $s \in S$ and $a(t) \gg a'(t)$ for some $t \in S$, then $a \succ a'$.

For any $\mathbf{n} \in \mathbb{N}^T$, we can associate to any $a \in A_{\mathbf{n}}$ an element in $\prod_{s=1}^{T} \mathbb{R}^{n_s}$, that is a can be viewed as a collection $(a(1), \cdots, a(T)) \in \prod_{s=1}^{T} \mathbb{R}^{n_s}$. We can therefore define the topology of $A_{\mathbf{n}}$ based on the product topology on $\prod_{s=1}^{T} \mathbb{R}^{n_s}$ to define the following notion of continuity.

Extended continuity. For any $\mathbf{n}, \hat{\mathbf{n}} \in \mathbb{N}^T$ and $a \in A_{\mathbf{n}}$, the sets $\{a' \in A_{\hat{\mathbf{n}}} | a \succsim a'\}$ and $\{a' \in A_{\hat{\mathbf{n}}} | a \precsim a'\}$ are closed.

All the orderings we will consider satisfy those two properties. We describe any ordering that does as "regular":

Definition 1 (Regularity). *A social welfare ordering $\succsim$ is regular if it satisfies Pareto and Extended continuity.*

4 Separable ethics

Social welfare economics has long debated the normative attractiveness of individual-level independence. The core of our paper is a weakening of this classic axiom that only assumes independence across species, while permitting non-separability among individuals within a species.

Species separability. For all a, a', $\hat{a}$, $\hat{a}' \in A$ and for all $\bar{S} \subset S$, $a_{\bar{S}}\hat{a} \succsim a'_{\bar{S}}\hat{a}$ if and only if $a_{\bar{S}}\hat{a}' \succsim a'_{\bar{S}}\hat{a}'$.

Given Species separability, the following axiom is normatively minimal (we can assess egalitarian paths within a species independently of what happens to other species):

Within-species egalitarian dominance. For all a, a', $\hat{a} \in A$,

(1) if there exists real numbers $x > y \geq 0$ and natural numbers $k > l$ and a species $s \in S$ such that $a(s) = x \cdot \mathbb{1}_k$ and $a'(s) = y \cdot \mathbb{1}_l$ then $a_s\hat{a} \succ a'_s\hat{a}$.

(2) if there exists real numbers $x < y \leq 0$ and natural numbers $k > l$ and a species $s \in S$ such that $a(s) = x \cdot \mathbb{1}_k$ and $a'(s) = y \cdot \mathbb{1}_l$ then $a_s\hat{a} \prec a'_s\hat{a}$.

The examples in Section 2 are violations of Species separability and Within-species egalitarian dominance. Species separability and Within-species egalitarian dominance are enough to obtain an additive representation, when $\succsim$ is regular. To state our first result, we need additional definitions. For any $k \in \mathbb{R}^k$, a function $F : \mathbb{R}^k \to \mathbb{R}$ is said to be non-decreasing if $F(u) \geq F(v)$ whenever $u, v \in \mathbb{R}^k$ and $u \geq v$. A function $F : \mathbb{R}^k \to \mathbb{R}$ is said to be normalized if $F(x \cdot \mathbb{1}_k) = x$ for all $x \in \mathbb{R}$.

Proposition 1. *If $\succsim$ is a regular social welfare ordering that satisfies Within-species egalitarian dominance and Species separability, then there exists continuous non-decreasing normalized functions $\Xi^s_n : \mathbb{R}^n \to \mathbb{R}$ and functions $V^s : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ that are continuous and increasing in their second argument such that, for all $a, a' \in A$*

$$a \succsim a' \iff \sum_{s\in S} V^s\Big(n\big(a(s)\big), \Xi^s_{n(a(s))}\big(a(s)\big)\Big) \geq \sum_{s\in S} V^s\Big(n\big(a'(s)\big), \Xi^s_{n(a'(s))}\big(a'(s)\big)\Big).$$

Furthermore $V^s(k, 0) = 0$ for all $k \in \mathbb{N}$ and $s \in S$, and for all $k > l$ and $s \in S$ $V^s(k, x) \geq V^s(l, x)$ if $x > 0$ and $V^s(k, x) \leq V^s(l, x)$ if $x < 0$.

Proof. See the Appendix. □

We interpret Ξ^s as a within-species equally-distributed-equivalent function that aggregates individual lifetime utilities into one representative utility. Notice that this function could be different for each species: utilitarian for cows, rank-dependent for pigs, prioritarian for cats.

Remark 1. *The proof of Proposition 1 depends on the existence of an indifferent expansion at zero for each species and on equivalent egalitarian distribution for larger populations: for a species with equal wellbeing level $x > 0$ (resp. $x < 0$) and size k, and for any $l > k$, there exists a level of wellbeing $y > 0$ (resp. $y < 0$) such that the population with l people and wellbeing y is equivalent to the original one. This is guaranteed by Within-species egalitarian dominance (and continuity). But we could use other principles, for instance a combination of Priority for lives worth living (Blackorby, Bossert and Donaldson, 2005) and Existence of a critical-level.*

5 Complementarity

Here we introduce a new axiom that assumes that, in allocating two species population sizes between two species, each with a different perfectly-egalitarian per-person welfare level, it is better to assign the larger population size to the species with better-off individuals.

Complementarity for species size and well-being. For all a, a', $\hat{a} \in A$, if there exists real numbers $x > y$ and natural numbers $k > l$ and two species $s, t \in S$ such that $a(s) = x \cdot \mathbb{1}_l$, $a(t) = y \cdot \mathbb{1}_k$, $a'(s) = x \cdot \mathbb{1}_k$ and $a'(t) = y \cdot \mathbb{1}_l$ then $a_{\{s,t\}}\hat{a} \prec a'_{\{s,t\}}\hat{a}$.

The property implies restrictions on the form of the V^s functions used in the additive representation of Proposition 1.

Proposition 2. *If $\succsim$ is a regular social welfare ordering that satisfies Within-species egalitarian dominance, Species separability and Complementarity for species size and well-being, then there exists functions $\Theta^s : \mathbb{R}_{++} \to \mathbb{R}$ and a function $V : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, continuous and increasing in its second argument, such that, for each $s \in S$, the function V^s of Proposition 1 can be written $V^s(k, x) = V(k, x) + \Theta^s(x)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$.*

Furthermore, for any $x > y$ and natural numbers $k > l$:

$$V(k, x) + V(l, y) > V(k, y) + V(l, x).$$

Proof. Given that $\succsim$ is a regular social welfare ordering that satisfies Within-species egalitarian dominance and species separability, we know by Prop 1 that there exists continuous non-decreasing normalized functions $\Xi_n^s : \mathbb{R}^n \to \mathbb{R}$ and functions $V^s : \mathbb{N}\times\mathbb{R} \to \mathbb{R}$, continuous and increasing in their second argument and with $V^s(n, 0) = 0$ for all $n \in \mathbb{N}$ and $s \in S$, such that, for all $a, a' \in A$

$$a \succsim a' \iff \sum_{s\in S} V^s\Big(n\big(a(s)\big), \Xi^s_{n(a(s))}\big(a(s)\big)\Big) \geq \sum_{s\in S} V^s\Big(n\big(a'(s)\big), \Xi^s_{n(a'(s))}\big(a'(s)\big)\Big).$$

Let us prove that for all $s \in S$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, we have $V^s(n, x) - V^1(n, x) = V^s(1, x) - V^1(1, x)$. Suppose that it is not the case and for instance that $V^s(n, x) - V^1(n, x) > V^s(1, x) - V^1(1, x)$ for some $s \in S$, $n \in \mathbb{N}$ and $x \in \mathbb{R}$.[1] Then, by continuity of V^s in its second argument, there exists $\varepsilon \in \mathbb{R}_{++}$ small enough so that $V^s(n, x - \varepsilon) - V^1(n, x) > V^s(1, x - \varepsilon) - V^1(1, x)$. Let a, a', $\hat{a} \in A$, be such that $a(s) = (x - \varepsilon) \cdot \mathbb{1}_n$, $a(1) = x \cdot \mathbb{1}_1$, $a'(s) = (x - \varepsilon) \cdot \mathbb{1}_1$ and $a'(1) = x \cdot \mathbb{1}_n$. $V^s(n, x - \varepsilon) + V^1(1, x) > V^s(1, x - \varepsilon) + V^1(n, x)$ implies $a_{\{s,t\}}\hat{a} \succ a'_{\{s,t\}}\hat{a}$, a violation of Complementarity for species size and well-being.

Therefore, for all $s \in S$, $n \in \mathbb{N}$, $x \in \mathbb{R}$, we have $V^s(n, x) = V^1(n, x) + V^s(1, x) - V^1(1, x) = V(n, x) + \Theta^s(x)$, where $\Theta^s(x) = V^s(1, x) - V^1(1, x)$ for all $x \in \mathbb{R}$.

Remark that using the representation above, using the definition of Complementarity for species size and well-being, we must have

$$V(k, x) + V(l, y) > V(k, y) + V(l, x)$$

for all $x > y$ and natural numbers $k > l$. □

This complementarity axiom has the effect of ruling out number-insensitive species-specific representations of the form $V^s(n, \Xi^s) = \Xi^s$, such as, for example, within-species average utilitarianism or maximin.

6 Replication invariance

For any l, $k \in \mathbb{N}$ and any vector $u \in \mathbb{R}^l$ we say that $v \in \mathbb{R}^{kl}$ is a k-replica of u if for all $i \in \{1, \cdots, l\}$ and $m \in \{1, \cdots, k\}$ we have $v_{(i-1)k+m} = u_i$. We denote $k \star u$ a k-replica of u. Similar, for any $a \in A$ we say that a' is a k-replica of a, denoted $k \star a$, if for each $s \in S$ we have $a'(s) = k \star a(s)$.

We introduce the following property, familiar in the population ethics literature, that guarantees that judgements are invariant to replications.

[1] The case $V^s(n, x) - V^1(n, x) < V^s(1, x) - V^1(1, x)$ can be dealt with similarly.

Strong replication invariance. For all a, $a' \in A$, for all $k \in \mathbb{N}$, $a \succsim a'$ if and only if $k \star a \succsim k \star a'$.

With this property we obtain the following characterization.

Proposition 3. *If $\succsim$ is a regular social welfare ordering that satisfies Within-species egalitarian dominance, Species separability and Strong replication invariance, then there exists $\alpha \geq 0$ and continuous and increasing function $\Psi^s : U \to \mathbb{R}$ such that $\Psi^s(0) = 0$ and $\Psi^s(u) = \Psi^s(k \star u)$ for all $k \in \mathbb{N}$ and $u \in U$, such that,*

$$a \succsim a' \Longleftrightarrow \sum_{s=1}^{S} \big[n(a(s))\big]^{\alpha} \Psi^s(a(s)) \geq \sum_{s=1}^{S} \big[n('a(s))\big]^{\alpha} \Psi^s(a'(s)).$$

If in addition $\succsim$ satisfies Complementarity for species size and well-being, then $\alpha > 0$ and there exists a continuous and increasing function $F : \mathbb{R} \to \mathbb{R}$ such that $F(0) = 0$ and for each $s \in S$ a continuous non-decreasing normalized functions $\Xi^s : \mathbb{R}^n \to \mathbb{R}$ such that $\Xi^s(u) = \Xi^s(k \star u)$ for all $k \in \mathbb{N}$ and $u \in U$ and $\Psi^s(u) = F \circ \Xi^s(u)$ for all $s \in S$ and $u \in U$.

Proof. See the Appendix. □

The social orderings characterized in Proposition 3 are the sum of the social valuation for each species. This social value for a species is itself the product of population size to the power $\alpha \geq 0$ and of a function of an equally-distributed equivalent for the species. This form is reminiscent of number-dampened utilitarian social welfare criteria proposed by Ng (1989). Specifically, Ng (1989) proposed a value function $V(u) = \frac{f(n(u))}{n(u)} \sum_{i \in N(u)} u_i$. Here we have $f(n(u) = n(u)^{\alpha}$: the case $\alpha = 1$ is similar to the so called total utilitarian case, while $\alpha = 0$ is similar to the so called average utilitarian case. The difference is that equally-distributed equivalent is not necessarily average utility.

Remark also that the addition of Complementarity for species size and well-being rules out the average case.

7 Anonymity

The last step in our characterization of generalized total utilitarianism is to assume individual-level anonymity. Implicitly, this axiom assumes that lifetime utilities are not only interpersonally comparable but also are comparable across species. If so, it is normatively attractive to assume that it does not matter, for example, whether a life of 3 is lived by a chimpanzee and a life of 4 is lived by a dog or the other way around.

Anonymity. For all $\mathbf{n} \in \mathbb{N}^T$ and all a, a', $\hat{a} \in A_{\mathbf{n}}$, if there exists two species $s, t \in S$ and $i \in \{1, \cdots, n_s\}$, $j \in \{1, \cdots, n_t\}$, such that $a_i(s) = a'_j(t)$, $a'_i(s) = a_j(t)$, $a_k(s) = a'_k(s)$ for all $k \neq i$ and $a_\ell(t) = a'_\ell(t)$ for all $\ell \neq j$, then $a_{\{s,t\}}\hat{a} \sim a'_{\{s,t\}}\hat{a}$.

Proposition 4. *If $\succsim$ is a regular social welfare ordering that satisfies Within-species egalitarian dominance, Species separability and Anonymity, then there exists a continuous and increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(0) = 0$ and, for all a $a' \in A$,*

$$a \succsim a' \Longleftrightarrow \sum_{s=1}^{S} \sum_{i=1}^{n(a(s))} \phi(a_i(s)) \geq \sum_{s=1}^{S} \sum_{i=1}^{n(a'(s))} \phi(a'_i(s)).$$

Proof. See the Appendix. □

Although none of our axioms directly assume individual-level independence, the intuition of the proof is that individual-level anonymity can be combined with species-level separability to construct individual-level independence, by hypothetically cycling people in and out of alternative species.

Note that contrary to our other characterization results, we obtain a very specific form of the value function for each species, namely a generalized total utilitarian one with a critical-level equal to zero. Although the other results allow for variable critical-level within a species this is not true when Anonymity is added. Therefore, the combination of Anonymity and Species separability has strong normative implications.

8 Discussion of possible extensions

8.1 Weakening separability or continuity

More social welfare functions become possible if we weaken Species separability to require separability only when one species is concerned.

Weak species separability. For all a, a', $\hat{a}$, $\hat{a}' \in A$ and for all $s \in S$, $a_s\hat{a} \succsim a'_s\hat{a}$ if and only if $a_s\hat{a}' \succsim a'_s\hat{a}'$.

In that case, we can find other social welfare criteria that satisfy Complementarity for species size and well-being, Within-species egalitarian dominance and Strong replication invariance. A first example uses a rank-dependent aggregator across species. Let Π be the set of permutations from S to S.

Example 1: Rank-dependence across species. Let $\beta_1 > \cdots > \beta_n$, preferences $\succsim$ are represented by the following social welfare function:

$$W(a) = \max_{\pi} \sum_{s \in S} \beta_{\pi(s)} \times \left[n\big(a(s)\big]^{\alpha} \times F\Big(\Xi^s\big(a_i(s)_{i \in N(a(s))}\big)\Big)$$

with F any continuous and increasing functions, and $\Xi^s : \mathbb{R}^n \to \mathbb{R}$ continuous non-decreasing normalized functions such that $\Xi^s(u) = \Xi^s(k \star u)$ for all $k \in \mathbb{N}$ and $u \in U$.

The rank-dependent aggregator assigns the highest weight β_1 to the species with highest welfare as measured by $\left[n\big(a(s)\big]^{\alpha} \times F\Big(\Xi^s\big(a_i(s)_{i \in N(a(s))}\big)\Big)$, and generality assigns weights in decreasing order of welfare. This is exactly the opposite of the standard rank-dependent case where we give more weight to worse off people. This suggests an anti-egalitarian feature of the aggregation. We conjecture that this anti-egalitarian feature is embedded in Complementarity for species size and well-being (we prefer to give more to better off species).

It is also possible to weaken Extended continuity. using a leximax aggregator, we could satisfy Species separability without additivity. For a vector $v = (v_1, \cdots, v_{|S|}) \in \mathbb{R}^{|S|}$, we denote $\tilde{v} \in \mathbb{R}^{|S|}$ the re-ordering of v in a non-increasing fashion, so that there exists $\pi \in \Pi$ such that $\tilde{v}_s = v_{\pi(s)}$ for all $s = 1, \cdots, |S|$ and $\tilde{v}_s \geq \tilde{v}_{s+1}$ for all $s = 1, \cdots, |S| - 1$. For two vector $v, v' \in \mathbb{R}^{|S|}$, we write $v >_{\text{lexmax}} v'$ if there is $s \in \{1, \cdots, |S|\}$ such that $\tilde{v}_t = \tilde{v'}_t$ for all $t = 1, \cdots, s-1$ and $\tilde{v}_s > \tilde{v'}_s$. We write $v =_{\text{lexmax}} v'$ if $\tilde{v} = \tilde{v}'$. And we write $v \geq_{\text{lexmax}} v'$ if either $v >_{\text{lexmax}} v'$ or $v =_{\text{lexmax}} v'$.

Example 2: Leximax across species. There exist continuous and increasing function F, and continuous non-decreasing normalized functions Ξ_s such that, for all $a, a' \in A$, $a \succsim a'$ if and only if

$$\left(\left[n\big(a(s)\big]^{\alpha} \times F\Big(\Xi^s\big(a_i(s)_{i \in N(a(s))}\big)\Big)\right)_{s \in S} \geq_{\text{lexmax}} \left(\left[n\big(a'(s)\big]^{\alpha} \times F\Big(\Xi^s\big(a'_i(s)_{i \in N(a'(s))}\big)\Big)\right)_{s \in S}.$$

8.2 Biodiversity

Between now and the SITE conference, we will explore extensions that represent measures of biodiversity. Interestingly, if both welfare and biodiversity are valued, then they would trade off against one another.

For example, one measure in the biodiversity literature is "evenness," which prefers an even number of members of different species (two cats and two dogs is better than three cats and one dog, or vice-versa). A multi-species social welfare function that valued both evenness and welfare would, in some cases, prefer an animal to be created that, on welfarist grounds alone, would live a life too full of suffering to be worth living, because the impersonal benefit of greater evenness outweighs the cost to the added individual of the suffering-filled, avoidable life. This example may strike some readers as reflecting another argument against non-welfarist biodiversity and in favor of cross-species separability.

8.3 Climate change

Between now and the SITE conference, we will show how our social welfare functions can be used as the objective function in standard climate-economy models to evaluate climate policies. An interesting implication is that climate change is expected to *increase* the size of some species, such as some insects. If there are enough insects, for example, and if their lives are valuable enough, it is possible, in principle, for these benefits to outweigh some other costs of climate change.

A Appendix

A.1 Preliminary

Let E be a set of factors and for each $e \in E$ X_e be an open connected and separable space. Let $\succsim$ be a complete and transitive relation on $\prod_{e\in E} X_e$. An element of $\mathbf{x} \in \prod_{e\in E} X_e$ is a collection $\mathbf{x} = (x_e)_{e\in E}$. Let $J \subset N$. We say that J is $\succsim$-separable if for any $\mathbf{x}$, $\hat{\mathbf{x}}$, $\mathbf{y}$ and $\hat{\mathbf{y}} \in \prod_{e\in E} X_e$ such that (i) $x_e = \hat{x}_e$ and $y_e = \hat{y}_e$ for all $e \in J$; and (ii) $x_{e'} = y_{e'}$ and $\hat{x}_{e'} = \hat{y}_{e'}$ for all $e' \in (E \setminus J)$; we have $\mathbf{x} \succsim \mathbf{y} \iff \hat{\mathbf{x}} \succsim \hat{\mathbf{y}}$. We say that J is strictly $\succsim$-essential if, for any $\mathbf{y} \in \prod_{e\in E} X_e$, there exist $\mathbf{x}$, $\hat{\mathbf{x}} \in \prod_{e\in E} X_e$ such that $y_e = x_e = \hat{x}_e$ for all $e \in (E \setminus J)$ but $\mathbf{x} \succ \hat{\mathbf{x}}$.

We say that $\succsim$ is totally separable if every subset $J \subset N$ is $\succsim$-separable. We have the following well-known result.

Lemma 1 (Debreu, 1960). *If $\succsim$ is a continuous, totally separable preference order on $\prod_{e\in E} X_e$, and every coordinate e is strictly $\succsim$-essential, then $\succsim$ has a fully additive utility representation: there exists continuous functions $\phi_e : X_e \to \mathbb{R}$ such that*

$$\mathbf{x} \succsim \mathbf{y} \iff \sum_{e\in E} \phi_e(x_e) \geq \sum_{e\in E} \phi_e(y_e).$$

Proof. See Theorem 3 in Debreu (1960). □

We also have the following well-know result.

Lemma 2 (Gorman, 1968). *If $\succsim$ be a continuous preference order on $\prod_{e\in E} X_e$. Let J, $K \subset E$ be two $\succsim$-separable subsets, such that $J \cap K \neq \emptyset$. Suppose that J, K, and $J \cap K$ are all strictly $\succsim$-essential. Then:*

1. *$J \cup K$ is $\succsim$-separable.*

2. *$J \cap K$ is $\succsim$-separable.*

Proof. See Gorman (1968) for the initial statement and Stengel (1993) for a general proof. □

A.2 Proof of Proposition 1

Step 1: An additive representation for each $\mathbf{n} \in \mathbb{N}^T$. For each $\mathbf{n} \in \mathbb{N}^T$, let us define the ordering $\succsim_{\mathbf{n}}$ on $\prod_{s=1}^{T} \mathbb{R}^{n_s}$ as follows. An element $\mathbf{u} \in \prod_{s=1}^{T} \mathbb{R}^{n_s}$ is $\mathbf{u} = (u^1, \cdots, u^T)$, where $u^s = (u^s_1, \cdots, u^s_{n_s}) \in \mathbb{R}^{n_s}$ for each $s \in S$. For all $\mathbf{u}$, $\hat{\mathbf{u}} \in \prod_{s=1}^{T} \mathbb{R}^{n_s}$, we define that $\mathbf{u} \succsim_{\mathbf{n}} \hat{\mathbf{u}}$ if and only if there exist a and $\hat{a} \in A_{\mathbf{n}}$ such that for each $s \in S$ $a(s) = u^s$ and $\hat{a} = \hat{u}^s$, and $a \succsim \hat{a}$.

Given that $\succsim$ is complete, transitive and continuous, so is $\succsim_{\mathbf{n}}$. By species Separability, the T factors of $\prod_{s=1}^{T} \mathbb{R}^{n_s}$ are $\succsim_{\mathbf{n}}$-separable. By Pareto, each of the T factors of $\prod_{s=1}^{T} \mathbb{R}^{n_s}$ is $\succsim_{\mathbf{n}}$-essential. Hence, by Lemma 1 there exists T continuous functions $\phi_{\mathbf{n}}^s : \mathbb{R}^{n_s} \to \mathbb{R}$ such that, for all $\mathbf{u}$, $\hat{\mathbf{u}} \in \prod_{s=1}^{T} \mathbb{R}^{n_s}$:

$$\mathbf{u} \succsim_{\mathbf{n}} \hat{\mathbf{u}} \Longleftrightarrow \sum_{s \in S} \phi_{\mathbf{n}}^s(u^s) \geq \sum_{s \in S} \phi_{\mathbf{n}}^s(\hat{u}^s).$$

By Pareto, each function $\phi_{\mathbf{n}}^s$ must be non-decreasing. By Pareto and continuity, for each $s \in S$ and each $u^s \in \mathbb{R}^{n_s}$, there exists a unique $x \in \mathbb{R}$ such that $\phi_{\mathbf{n}}^s(u^s) = \phi_{\mathbf{n}}^s(x \cdot \mathbb{1}_{n_s})$; and for all $x > y$: $\phi_{\mathbf{n}}^s(x \cdot \mathbb{1}_{n_s}) > \phi_{\mathbf{n}}^s(y \cdot \mathbb{1}_{n_s})$. We can find a continuous and increasing function $f_{\mathbf{n}}^s : \phi_{\mathbf{n}}^s(\mathbb{R}) \to \mathbb{R}$ such that $f_{\mathbf{n}}^s(\phi_{\mathbf{n}}^s(x \cdot \mathbb{1}_{n_s})) = x$ for all $x \in \mathbb{R}$. Denote $\Xi_{\mathbf{n}}^s : \mathbb{R}^{n_s} \to \mathbb{R}$ the function $\Xi_{\mathbf{n}}^s = f_{\mathbf{n}}^s \circ \phi_{\mathbf{n}}^s$: $\Xi_{\mathbf{n}}^s$ is normalized, continuous and non-decreasing. Denoting $\bar{V}_{\mathbf{n}}^s$ the inverse function of $f_{\mathbf{n}}^s$, we obtain that for any a and $\hat{a} \in A_{\mathbf{n}}$:

$$a \succsim \hat{a} \Longleftrightarrow \sum_{s \in S} \bar{V}_{\mathbf{n}}^s\Big(\Xi_{\mathbf{n}}^s\big(a(s)\big)\Big) \geq \sum_{s \in S} \bar{V}_{\mathbf{n}}^s\Big(\Xi_{\mathbf{n}}^s\big(\hat{a}(s)\big)\Big).$$

Step 2: A representation of $\succsim$. Let us define the ordering $\hat{\succsim}$ on $\mathbb{N}^T \times \mathbb{R}^T$ as follows: for any $\mathbf{n}$, $\hat{\mathbf{n}} \in \mathbb{N}^T$ and for any $\mathbf{x}$, $\hat{\mathbf{x}} \in \mathbb{R}^T$, $(\mathbf{n}, \mathbf{x}) \hat{\succsim} (\hat{\mathbf{n}}, \hat{\mathbf{x}})$ if and only if there exist a, $\hat{a} \in A$ such that $a(s) = x_s \cdot \mathbb{1}_{n_s}$ and $\hat{a}(s) = \hat{x}_s \cdot \mathbb{1}_{\hat{n}_s}$ for all $s \in S$ and $a \succsim \hat{a}$.

Using Pareto and Extended Continuity, we can mimic the proof of Theorem 2 in Blackorby, Bossert and Donaldson (1984) to show that there exists a function $W : \mathbb{N}^T \times \mathbb{R}^T \to \mathbb{R}$ such that for any $\mathbf{x}$, $\hat{\mathbf{x}} \in \mathbb{R}^T$, $(\mathbf{n}, \mathbf{x}) \hat{\succsim} (\hat{\mathbf{n}}, \hat{\mathbf{x}})$ if and only $W(\mathbf{n}, \mathbf{x}) \geq W(\hat{\mathbf{n}}, \hat{\mathbf{x}})$, and W is continuous and increasing in the vector in $\mathbb{R}^T$.

By Step 1 and the definition of $\hat{\succsim}$, we know that for any $\mathbf{n} \in \mathbb{N}^T$, and any $\mathbf{x}$, $\hat{\mathbf{x}} \in \mathbb{R}^T$, $(\mathbf{n}, \mathbf{x}) \hat{\succsim} (\hat{\mathbf{n}}, \hat{\mathbf{x}})$ if and only $\sum_{s \in S} \bar{V}_{\mathbf{n}}^s(x_s) \geq \sum_{s \in S} \bar{V}_{\mathbf{n}}^s(\hat{x}_s)$. So, for each $\mathbf{n} \in \mathbb{N}^T$ there must exist an increasing function $\bar{\Phi}_{\mathbf{n}}$ such that for all $\mathbf{x} \in \mathbb{R}^T$:

$$W(\mathbf{n}, \mathbf{x}) = \bar{\Phi}_{\mathbf{n}} \left(\sum_{s \in S} \bar{V}_{\mathbf{n}}^s(x_s) \right).$$

Given that, W is continuous and increasing in the vector in $\mathbb{R}^T$, $\bar{\Phi}_{\mathbf{n}}$ must be continuous.

Writing $\tilde{V}(x_s) = \bar{V}^s_{\mathbf{n}}(x_s) - \bar{V}^s_{\mathbf{n}}(0)$ and $\tilde{\Phi}_{\mathbf{n}}(z) = \bar{\Phi}_{\mathbf{n}}\left(z + \sum_{s\in S} \bar{V}^s_{\mathbf{n}}(0)\right)$, we thus have:

$$W(\mathbf{n}, \mathbf{x}) = \tilde{\Phi}_{\mathbf{n}}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}}(x_s)\right).$$

By Step 1, we also know that for any $\mathbf{n}$ and $a \in A_{\mathbf{n}}$, if $a' \in A_{\mathbf{n}}$ such that $a'(s) = \Xi^s_{\mathbf{n}}\big(\hat{a}(s)\big) \cdot \mathbb{1}_{n(a(s))}$ for all $s \in S$ then $a \sim a'$. So we end up with the following form for the social ordering: for all a, $\hat{a} \in A$,

$$a \succsim \hat{a} \Longleftrightarrow \tilde{\Phi}_{\mathbf{n}(a)}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}(a)}\Big(\Xi^s_{\mathbf{n}(a)}\big(a(s)\big)\Big)\right) \geq \tilde{\Phi}_{\mathbf{n}(\hat{a})}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}(\hat{a})}\Big(\Xi^s_{\mathbf{n}(\hat{a})}\big(\hat{a}(s)\big)\Big)\right)$$

Denote $a^0 \in A$ the allocation such that $a^0(s) = 0$ for all $s \in S$. For any $s \in S$, for any $k \in \mathbb{N}$, consider a, $\hat{a} \in A$ such that $n(a(s)) = n(\hat{a}(s)) = k$, and $a(t) = \hat{a}(t)$ for all $t \neq s$. By species separability, $a \succsim \hat{a}$ if and only if $a_s a^0 \succsim \hat{a}_s a^0$. By our representation above, this means that $\Xi^s_{\mathbf{n}(a)}\big(a(s)\big) \geq \Xi^s_{\mathbf{n}(\hat{a})}\big(\hat{a}(s)\big)$ if and only if $\Xi^s_{\mathbf{n}(a_s a^0)}\big(a(s)\big) \geq \Xi^s_{\mathbf{n}(\hat{a}_s a^0)}\big(\hat{a}(s)\big)$. Denoting $\Xi^s_{\mathbf{n}(a)}$ by Ξ^s_k when $n(a(s)) = k$ and $n(a(t)) = 1$ for all $t \neq s$, we get that $\Xi^s_{\mathbf{n}(a)}(x) = \Xi^s_k(x)$ for all $x \in \mathbb{R}$ whenever $n(a(s)) = k$ (because each $\Xi^s_{\mathbf{n}(a)}$ is normalized).

To sum up, there exists continuous non-decreasing normalized functions $\Xi^s_n : \mathbb{R}^n \to \mathbb{R}$, functions $\tilde{V}_{\mathbf{n}} : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, continuous and increasing in their second argument, and continuous and increasing functions $\tilde{\Phi}_{\mathbf{n}(a)}$ such that for all a, $\hat{a} \in A$,

$$a \succsim \hat{a} \Longleftrightarrow \tilde{\Phi}_{\mathbf{n}(a)}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}(a)}\Big(\Xi^s_{n(a(s))}\big(a(s)\big)\Big)\right) \geq \tilde{\Phi}_{\mathbf{n}(\hat{a})}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}(\hat{a})}\Big(\Xi^s_{n(\hat{a}(s))}\big(\hat{a}(s)\big)\Big)\right). \quad \text{(A.1)}$$

Step 3: Indifferent expansion at 0. Consider any $s \in S$, and any positive integers $l > m$. For any $a' \in A$ and any $\varepsilon > 0$, Within-species egalitarian dominance implies that $a_s a' \succ \hat{a}_s a'$ whenever $a(s) = \varepsilon \cdot \mathbb{1}_l$ and $\hat{a}(s) = 0 \cdot \mathbb{1}_m$. By Extended continuity, this implies that $a_s a' \succsim \hat{a}_s a'$ whenever $a(s) = 0 \cdot \mathbb{1}_l$ and $a(s) = 0 \cdot \mathbb{1}_m$.

Symmetrically, for any $s \in S$, any positive integers $l > m$, any $a' \in A$ and any $\varepsilon < 0$, Within-species egalitarian dominance implies that $a_s a' \prec \hat{a}_s a'$ whenever $a(s) = \varepsilon \cdot \mathbb{1}_l$ and $\hat{a}(s) = 0 \cdot \mathbb{1}_m$. By Extended continuity, this implies that $a_s a' \precsim \hat{a}_s a'$ whenever $a(s) = 0 \cdot \mathbb{1}_l$ and $a(s) = 0 \cdot \mathbb{1}_m$.

Combining those results, this implies that for any $s \in S$, any positive integers $l > m$, any $a' \in A$, and any a and $\hat{a}$ such that $a(s) = 0 \cdot \mathbb{1}_l$ and $\hat{a}(s) = 0 \cdot \mathbb{1}_m$, we have $a_s a' \sim \hat{a}_s a'$.

By a similar reasoning, we obtain that, for each $x \in \mathbb{R}_{++}$ (resp. $x \in \mathbb{R}_{--}$) there exists $y \in \mathbb{R}_{++}$ such that $y \leq x$ (resp. $y \in \mathbb{R}_{--}$ such that $y \geq x$) such that, for any positive integers $l > m$, any $a' \in A$, and any a and $\hat{a}$ such that $a(s) = y \cdot \mathbb{1}_l$ and $\hat{a}(s) = x \cdot \mathbb{1}_m$, we have $a_s a' \sim \hat{a}_s a'$.

Step 4: Link between the $V^s_{\mathbf{n}}$ representations. For any $\mathbf{n} \in \mathbb{N}^T$, denote $a^0_{\mathbf{n}} \in A$ the allocation such that $a^0_{\mathbf{n}}(s) = 0 \cdot \mathbb{1}_{n_s}$ for all $s \in S$.

Consider any $s \in S$, k, $l \in \mathbb{N}$ with $l \geq k$ and $\mathbf{n}$, $\hat{\mathbf{n}} \in \mathbb{N}^T$ such that $n_s = k$, $\hat{n}_s = l$. By Step 3 and species separability, we know that for each $x \in \mathbb{R}$ there exists $y \in \mathbb{R}$ such that, if a, $\hat{a} \in A$ are such that $a(s) = x \cdot \mathbb{1}_k$ and $\hat{a}(s) = y \cdot \mathbb{1}_l$, then $a_s a^0_{\mathbf{n}} \sim \hat{a}_s a^0_{\hat{\mathbf{n}}}$. By our representation above, we must have:

$$\tilde{\Phi}_{\mathbf{n}}\Big(\tilde{V}^s_{\mathbf{n}}(x)\Big) = \tilde{\Phi}_{\hat{\mathbf{n}}}\Big(\tilde{V}^s_{\hat{\mathbf{n}}}(y)\Big). \tag{A.2}$$

In particular if $l = k$, we must have $x = y$ for all $x \in \mathbb{R}$.

Consider any $\mathbf{n}$, $\mathbf{n}' \in N^T$ such that $\mathbf{n}' > \mathbf{n}$. For any $\mathbf{x} \in \mathbb{R}^T$, by a reasoning similar to the one above, there exists $\hat{\mathbf{x}} \in \mathbb{R}^T$ such that for each $s \in S$ we have $\tilde{\Phi}_{\mathbf{n}}\Big(\tilde{V}^s_{\mathbf{n}}(x_s)\Big) = \tilde{\Phi}_{\mathbf{n}'}\Big(\tilde{V}^s_{\mathbf{n}'}(\hat{x}_s)\Big)$ and, if a, $\hat{a} \in A$ are defined by $a(s) = x_s \cdot \mathbb{1}_{n_s}$ and $\hat{a}(s) = \hat{x}_s \cdot \mathbb{1}_{n'_s}$ then $a \sim \hat{a}$. By our representation above, this implies that:

$$\tilde{\Phi}_{\mathbf{n}}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}}(x_s)\right) = \tilde{\Phi}_{\mathbf{n}'}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}'}(\hat{x}_s)\right).$$

We can write it:

$$\tilde{\Phi}^{-1}_{\mathbf{n}'} \circ \tilde{\Phi}_{\mathbf{n}}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}}(x_s)\right) = \sum_{s\in S} \tilde{\Phi}^{-1}_{\mathbf{n}'} \circ \tilde{\Phi}_{\mathbf{n}}\Big(\tilde{V}^s_{\mathbf{n}}(x_s)\Big).$$

Given that each $\tilde{V}^s_{\mathbf{n}}$ is continuous, $\tilde{V}^s_{\mathbf{n}}(\mathbb{R}) = X^s_{\mathbf{n}}$ is a connected open set, and denoting $F_{\mathbf{n},\mathbf{n}'} := \tilde{\Phi}^{-1}_{\mathbf{n}'} \circ \tilde{\Phi}_{\mathbf{n}}$ we obtain the following functional equation on $\prod_{s=1}^T X^s_{\mathbf{n}}$:

$$F_{\mathbf{n},\mathbf{n}'}\left(\sum_{s=1}^T x_s\right) = \sum_{s=1}^T F_{\mathbf{n},\mathbf{n}'}(x_s).$$

This is a Cauchy function equation. Given that $F_{\mathbf{n},\mathbf{n}'}$ is continuous, we know (Aczél, 1966, Chap. 2) that $F_{\mathbf{n},\mathbf{n}'}$ must be linear. Hence, there exists $\gamma_{\mathbf{n},\mathbf{n}'}$ such that

$$F_{\mathbf{n},\mathbf{n}'}(x) = \gamma_{\mathbf{n},\mathbf{n}'} \times x \tag{A.3}$$

for all $x \in X_{\mathbf{n}}$, where $X_{\mathbf{n}} = \{x \in \mathbb{R} | \exists \mathbf{x} \in \prod_{s=1}^T X^s_{\mathbf{n}}, x = \sum_{s=1}^T x_s\}$. Remark that $X^s_{\mathbf{n}} \subset X_{\mathbf{n}}$

for each $s \in S$.

Let $\mathbf{n^1} \in \mathbb{N}^T$ be $\mathbf{n^1} = \mathbb{1}_T$ (a population with 1 individual in each specie). For each $s \in S$ and $x \in \mathbb{R}$, Equation A.2 tells us that for each $\mathbf{n} \in \mathbb{N}^T$ there exists $y \in \mathbb{R}$ such that $\tilde{\Phi}_{\mathbf{n^1}}\left(\tilde{V}^s_{\mathbf{n^1}}(x)\right) = \tilde{\Phi}_{\mathbf{n}}\left(\tilde{V}^s_{\mathbf{n}}(y)\right)$. By the result in the previous paragraph, we have that $\tilde{V}^s_{\mathbf{n}}(y) = \gamma_{\mathbf{n^1},\mathbf{n}} \times \tilde{V}^s_{\mathbf{n^1}}(x)$. Let us denote $Y^s_{\mathbf{n}} = \{y \in \mathbb{R} | \exists x \in \mathbb{R}, \tilde{V}^s_{\mathbf{n}}(y) = \gamma_{\mathbf{n^1},\mathbf{n}} \times \tilde{V}^s_{\mathbf{n^1}}(x)\}$. Again, this is an open connected subset in $\mathbb{R}$ that includes 0.

Denote $\mathbf{n^{k,s}} \in \mathbb{N}^T$ such that $n^{k,s}_s = k$ and $n^{k,s}_t = 1$ for all $t \neq s$. For each $s \in S$, $x \in \mathbb{R}$, $k \in \mathbb{N}$, $\mathbf{n}$ such that $n_s = k$, Equations A.2 and A.3 imply that:

$$\tilde{V}^s_{\mathbf{n}}(x) = \gamma_{\mathbf{n^{k,s}},\mathbf{n}} \times \tilde{V}^s_{\mathbf{n^{k,s}}}(x)$$

for all $x \in \mathbb{R}$. Furthermore, for any $x \in \mathbb{R}$, and any $\mathbf{n}' \geq \mathbf{n}$, Equations A.2 and A.3 imply that there exists $y \in \mathbb{R}$ such that:

$$\tilde{V}^s_{\mathbf{n}'}(y) = \gamma_{\mathbf{n},\mathbf{n}'} \times \tilde{V}^s_{\mathbf{n}}(x) = \gamma_{\mathbf{n},\mathbf{n}'} \times \gamma_{\mathbf{n^{k,s}},\mathbf{n}} \times \tilde{V}^s_{\mathbf{n^{k,s}}}(x), \tag{A.4}$$

and such that, if $a, \hat{a} \in A$ are such that $a(s) = x \cdot \mathbb{1}_k$ and $\hat{a}(s) = y \cdot \mathbb{1}_{\mathbf{n}'_s}$, then $a_s a^0_{\mathbf{n}} \sim \hat{a}_s a^0_{\hat{\mathbf{n}}}$.

Gathering those results, we find that, for each $k \in \mathbb{N}$, any $\mathbf{n}, \mathbf{n}' \in \mathbb{N}^T$ such that $n_s = k$ and $\mathbf{n}' \geq \mathbf{n}$, for any $y \in Y^s_{\mathbf{n^{k,s}}}$, there exists $x \in \mathbb{R}$ and $z \in \mathbb{R}$ such that $\tilde{V}^s_{\mathbf{n}}(y) = \gamma_{\mathbf{n^1},\mathbf{n^{k,s}}} \times \tilde{V}^s_{\mathbf{n^1}}(x)$, $\tilde{V}^s_{\mathbf{n}'}(z) = \gamma_{\mathbf{n},\mathbf{n}'} \times \gamma_{\mathbf{n^{k,s}},\mathbf{n}} \times \tilde{V}^s_{\mathbf{n^{k,s}}}(y)$, so that, if $a, a', \hat{a} \in A$ are such that $a(s) = y \cdot \mathbb{1}_k$, $a'(s) = z \cdot \mathbb{1}_{n'_s}$ and $\hat{a}(s) = x$, then $a_s a^0_{\mathbf{n}} \sim a_s a^0_{\mathbf{n^1}} \sim a'_s a^0_{\mathbf{n}'}$ and $a_s a^0_{\mathbf{n}} \sim a_s a^0_{\mathbf{n^1}} \sim \hat{a}_s a^0_{\mathbf{n^1}}$. Thus, $a'_s a^0_{\mathbf{n}'} \sim \hat{a}_s a^0_{\mathbf{n^1}}$, and we must have $\tilde{V}^s_{\mathbf{n}'}(z) = \gamma_{\mathbf{n^1},\mathbf{n}'} \times \tilde{V}^s_{\mathbf{n^1}}(x)$. Hence:

$$\gamma_{\mathbf{n^1},\mathbf{n}'} \times \tilde{V}^s_{\mathbf{n^1}}(x) = \tilde{V}^s_{\mathbf{n}'}(z) = \gamma_{\mathbf{n},\mathbf{n}'} \times \gamma_{\mathbf{n^{k,s}},\mathbf{n}} \times \tilde{V}^s_{\mathbf{n^{k,s}}}(y) = \gamma_{\mathbf{n},\mathbf{n}'} \times \gamma_{\mathbf{n^{k,s}},\mathbf{n}} \times \gamma_{\mathbf{n^1},\mathbf{n^{k,s}}} \times \tilde{V}^s_{\mathbf{n^1}}(x).$$

We can write such an equation for any $x \in \mathbb{R}$. Therefore, we must have:

$$\gamma_{\mathbf{n},\mathbf{n}'} \times \gamma_{\mathbf{n^{k,s}},\mathbf{n}} = \frac{\gamma_{\mathbf{n^1},\mathbf{n}'}}{\gamma_{\mathbf{n^1},\mathbf{n^{k,s}}}}.$$

For any $\mathbf{n}' \in \mathbb{N}^T$, denote $\theta_{\mathbf{n}'} := \gamma_{\mathbf{n^1},\mathbf{n}'}$ and for any $k \in \mathbb{N}$ and $s \in S$, denote $V^s_k := \frac{1}{\gamma_{\mathbf{n^1},\mathbf{n^{k,s}}}} \times \tilde{V}^s_k$. We obtain that any $\mathbf{n}' \geq \mathbf{n}$, and for any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that:

$$\tilde{V}^s_{\mathbf{n}'}(y) = \theta_{\mathbf{n}'} \times V^s_k(x), \tag{A.5}$$

and such that, if $a, \hat{a} \in A$ are such that $a(s) = x \cdot \mathbb{1}_k$ and $\hat{a}(s) = y \cdot \mathbb{1}_{\mathbf{n}'_s}$, then $a_s a^0_{\mathbf{n}} \sim \hat{a}_s a^0_{\hat{\mathbf{n}}}$.

Step 5: Conclusion.

Consider any a and $\hat{a} \in A$. Let $\mathbf{n}' \in \mathbb{N}^T$ be such that $n'_s = \max\left\{n(a(s)), n(\hat{a}(s)\right\}$

for each $s \in S$. Using the representation in Equation A.1, let $\mathbf{x}$, $\hat{\mathbf{x}} \in \mathbb{R}^T$ be such that $x_s = \Xi^s_{n(a(s))}\big(a(s)\big)$ and $\hat{x}_s = \Xi^s_{n(\hat{a}(s))}\big(\hat{a}(s)\big)$ for each $s \in S$. Let $\mathbf{y}$, $\hat{\mathbf{y}} \in \mathbb{R}^T$ be such that $\tilde{V}^s_{\mathbf{n}'}(y_s) = \theta_{\mathbf{n}'} \times V^s_{n(a(s))}(x_s)$ and $\tilde{V}^s_{\mathbf{n}'}(y_s) = \theta_{\mathbf{n}'} \times V^s_{n(\hat{a}(s))}(\hat{x}_s)$ for each $s \in S$. Given that, for any $a \in A$ and $s \in S$, $a_s a^0_{\mathbf{n}} \sim \tilde{a}_s a^0_{\hat{\mathbf{n}}}$ when $\tilde{a}(s) = \Xi^s_{n(\hat{a}(s))}\big(\hat{a}(s)\big) \cdot \mathbb{1}_{n(a(s))}$, and given the result is Step 4 and species separability, if we define a' and $\hat{a}' \in A_{\mathbf{n}'}$ by $a'(s) = y_s \cdot \mathbb{1}_{n'_s}$ and $\hat{a}'(s) = \hat{y}_s \cdot \mathbb{1}_{n'_s}$ for each $s \in S$, we obtain $a \sim a'$ and $\hat{a} \sim \hat{a}'$.

Thus $a \succsim \hat{a} \iff a' \succsim \hat{a}'$. But using Equation A.1 and the definitions of a' and $\hat{a}'$:

$$\begin{aligned} a' \succsim \hat{a}' \iff & \tilde{\Phi}_{\mathbf{n}'}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}'}(y'_s)\right) \geq \tilde{\Phi}_{\mathbf{n}'}\left(\sum_{s\in S} \tilde{V}^s_{\mathbf{n}'}(\hat{y}'_s)\right) \\ \iff & \sum_{s\in S} \theta_{\mathbf{n}'} \times V^s_{n(a(s))}(x_s) \geq \sum_{s\in S} \theta_{\mathbf{n}'} \times V^s_{n(\hat{a}(s))}(\hat{x}_s) \\ \iff & \sum_{s\in S} V^s_{n(a(s))}\left(\Xi^s_{n(a(s))}\big(a(s)\big)\right) \geq \sum_{s\in S} V^s_{n(\hat{a}(s))}\left(\Xi^s_{n(\hat{a}(s))}\big(\hat{a}(s)\big)\right). \end{aligned}$$

A.3 Proof of Proposition 3

Given that $\succsim$ is a regular social welfare ordering that satisfies Within-species egalitarian dominance and species separability, we know by Prop 1 that there exists continuous non-decreasing normalized functions $\Xi^s_n : \mathbb{R}^n \to \mathbb{R}$ and functions $V^s : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, continuous and increasing in their second argument such that, for all $a, a' \in A$

$$a \succsim a' \iff \sum_{s\in S} V^s\Big(n\big(a(s)\big), \Xi^s_{n(a(s))}\big(a(s)\big)\Big) \geq \sum_{s\in S} V^s\Big(n\big(a'(s)\big), \Xi^s_{n(a'(s))}\big(a'(s)\big)\Big).$$

Furthermore $V^s(n, 0) = 0$ for all $n \in \mathbb{N}$ and $s \in S$, and for all $k > l$ and $s \in S$ $V^s(k, x) \geq V^s(l, x)$ if $x > 0$ and $V^s(k, x) \leq V^s(l, x)$ if $x < 0$.

Consider any $s, t \in S$, l, m and $k \in \mathbb{N}$ and x, $\hat{x}$, y and $\hat{y} \in \mathbb{R}$. Let a, a' and $\hat{a} \in A$ be such that $a(s) = x \cdot \mathbb{1}_l$, $\hat{a}(s) = \hat{x} \cdot \mathbb{1}_l$, $a(t) = y \cdot \mathbb{1}_m$, $\hat{a}(t) = \hat{y} \cdot \mathbb{1}_m$. By Strong replication invariance, $a_{\{s,t\}}\hat{a} \succsim a'_{\{s,t\}}\hat{a}$ if and only if $k \star a_{\{s,t\}} k \star \hat{a} \succsim k \star a'_{\{s,t\}} k \star \hat{a}$. By the representation above, this means that for all x, $\hat{x}$, y and $\hat{y} \in \mathbb{R}$:

$$V^s(l, x) + V^t(m, y) \geq V^s(l, \hat{x}) + V^t(m, \hat{y}) \iff V^s(kl, x) + V^t(km, y) \geq V^s(kl, \hat{x}) + V^t(km, \hat{y}).$$

By the unicity of additive representations up to a positive affine transformation (Debreu, 1960), there must exist $\gamma^{l,m}_k > 0$ and $\beta^{l,m}_k$ such that: $V^s(kl, x) = \gamma^{l,m}_k V^s(l, x) + \beta^{l,m}_k$ and $V^t(km, x) = \gamma^{l,m}_k V^t(m, x) + \beta^{l,m}_k$ for all $x \in \mathbb{R}$. Given that $V^s(n, 0) = 0$ for all $n \in \mathbb{N}$ and $s \in S$, we must have $\beta^{l,m}_k = 0$. Remark also that, if we pick $m' \neq m$, we should have by a similar reasoning $V^s(kl, x) = \gamma_k^{l,m'} V^s(l, x)$ for some $\gamma_k^{l,m'} > 0$ so that $\gamma_k^{l,m'} = \gamma_k^{l,m}$ for all $m, m' \in \mathbb{N}$. Similarly, if we pick $l' \neq l$, we should have $V^t(km, x) = \gamma_k^{l',m} Vt(m, x)$ for some $\gamma_k^{l',m} > 0$ so that $\gamma_k^{l',m} = \gamma_k^{l,m}$ for all $l, l' \in \mathbb{N}$.

Given that species s, $t \in S$ are taken arbitrarily and we can combine any pairs, and given that the constant $\gamma_k^{l,m}$ does not depend on l and m, we obtain that for any $s \in S$, any $x \in \mathbb{R}$ and any l and $k \in \mathbb{N}$ we have $V^s(kl, x) = \gamma_k^{1,1} V_s(l, x)$. Denote $\rho : \mathbb{N} \to \mathbb{R}$ the function such that $\rho(k) = \gamma_k^{1,1}$ for each $k \in \mathbb{N}$. Hence $V_s(kl, x) = \rho(k) V_s(l, x)$, which implies that $V_s(kl, x) = \rho(k)\rho_s(l) V^s(1, x)$ and also that $V^s(kl, x) = \rho(kl) V^s(1, x)$. Therefore $\rho(kl) = \rho(k)\rho(l)$ for all l and $k \in \mathbb{N}$. By Within-species egalitarian dominance, we must also have that ρ is non-decreasing. By Theorem A in Howe (1986), we know that there must exist $\alpha \in \mathbb{R}_+$ such that $\rho(k) = k^\alpha$. And therefore $V^s(k, x) = k^\alpha V^s(1, e)$ for all $k \in \mathbb{N}$ and $x \in \mathbb{R}$.

For any $s \in S$, $k \in \mathbb{N}$, u, $v \in U$ such that $n(u) = n(v)$, considering a, a' and $\hat{a} \in A$ such that $a(s) = u$ and $\hat{a}(s) = v$, Strong replication invariance implies that $a_s \hat{a} \succsim a'_s \hat{a}$ if and only if $k \star a_s k \star \hat{a} \succsim k \star a'_s k \star \hat{a}$. By the representation results, this means that:

$$V^s\Big(1, \Xi^s_{n(u)}(u)\Big) \geq V^s\Big(1, \Xi^s_{n(v)}(v)\Big) \iff V^s\Big(1, \Xi^s_{kn(u)}(k \star u)\Big) \geq V^s\Big(1, \Xi^s_{kn(v)}(k \star v)\Big),$$

for all u, $v \in U$ such that $n(u) = n(v)$. Denoting $\Psi^s : U \to \mathbb{R}$ such that $\Psi^s(u) = Vs\Big(1, \Xi^s_{n(u)}(u)\Big)$ for all $u \in U$, given that Ξ^s_n is normalized, we must have $\Psi^s(u) = \Psi^s(k \star u)$ for all $k \in \mathbb{N}$ and $u \in U$. By definition, we also have $\Psi^s(0) = 0$.

Last assume that $\succsim$ satisfies Complementarity for species size and well-being. By our result above we have $V^s(k, x) = k^\alpha V^s(1, x)$ for all $x \in \mathbb{R}$. By Proposition 2, we have $V^s(k, x) = V^1(k, x) + \Theta^s(x)$ for all $x \in \mathbb{R}$. Combining those results, we necessarily have $\Theta^s(x) = 0$ for all $x \in \mathbb{R}$.

A.4 Proof of Proposition 4

Step 1: A fully additive and symmetric representation for each $\mathbf{n} \in \mathbb{N}^T$.

Given that $\succsim$ is a regular social welfare ordering that satisfies Within-species egalitarian dominance and species separability, we know by Prop 1 that there exists continuous non-decreasing normalized functions $\Xi^s_n : \mathbb{R}^n \to \mathbb{R}$ and functions $V^s : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$, continuous and increasing in their second argument such that, for all $a, a' \in A$

$$a \succsim a' \iff \sum_{s \in S} V^s\Big(n\big(a(s)\big), \Xi^s_{n(a(s))}\big(a(s)\big)\Big) \geq \sum_{s \in S} V^s\Big(n\big(a'(s)\big), \Xi^s_{n(a'(s))}\big(a'(s)\big)\Big).$$

Furthermore $V^s(n,0) = 0$ for all $n \in \mathbb{N}$ and $s \in S$, and for all $k > l$ and $s \in S$ $V^s(k,x) \geq V^s(l,x)$ if $x > 0$ and $V^s(k,x) \leq V^s(l,x)$ if $x < 0$.

For $\mathbf{n} = 1 \cdot \mathbb{1}_T$, we thus have that for all $a, a' \in A_{1\cdot\mathbb{1}_T}$:

$$a \succsim a' \iff \sum_{s\in S} \phi^s(a(s)) \geq \sum_{s\in S} \phi^s(a'(s)),$$

with $\phi^s(x) = V^s(1,x)$ for all $x \in \mathbb{R}$. By Anonymity, the ϕ^s function must all be the same function ϕ, with $\phi(0) = 0$.

For each $\mathbf{n} \in \mathbb{N}^T$ such that $n_s \geq 2$ for some $s \in S$, let us denote $E(\mathbf{n}) = \{1, \cdots, \sum_{s\in S} n_s\}$, $X_e = \mathbb{R}$ for each $e \in E(\mathbf{n})$. Let us define the ordering $\tilde{\succsim}_{\mathbf{n}}$ on $\prod_{e\in E(\mathbf{n})} X_e$ as follows. We say that $a \in A_{\mathbf{n}}$ is associated to $\mathbf{x}$ if for each $s \in S$ and $i \in \{1, \cdots, n_s\}$ we have $a_i(s) = x_e$ where $e = \sum_{t=1}^{s-1} n_t + i$.[2] For each $\mathbf{x}, \mathbf{y} \in \prod_{e\in E(\mathbf{n})} X_e$, we define that $\mathbf{x} \tilde{\succsim}_{\mathbf{n}} \mathbf{y}$ if there exist a and $\hat{a} \in A_{\mathbf{n}}$ such that a is associated to $\mathbf{x}$, $\hat{a}$ is associated to $\mathbf{y}$ and $a \succsim \hat{a}$.

By Anonymity, a permutation of two components in e and e' is indifferent for $\tilde{\succsim}_{\mathbf{n}}$ provided there exists s, $t \in S$ with $s \neq t$, $i \in \{1, \cdots, n_s\}$ and $j \in \{1, \cdots, n_t\}$ such that $e = \sum_{t=1}^{s-1} n_t + i$ and $e' = \sum_{t=1}^{t-1} n_t + j$. But by chains of such permutations (and transitivity of $\tilde{\succsim}_{\mathbf{n}}$), it is clear that the permutation of any two components if indifferent for $\tilde{\succsim}_{\mathbf{n}}$. Also, by chains of two components permutations, any permutations (of more than two components) is indifferent for $\tilde{\succsim}_{\mathbf{n}}$.

For each $\mathbf{n} \in \mathbb{N}^T$, by species separablity, any subset of components $E^S(\mathbf{n})\{\sum_{t=1}^{s-1} n_t + 1, \cdots, \sum_{t=1}^{s} n_t\} \subset E(\mathbf{n})$ with $s \in S$ is $\tilde{\succsim}_{\mathbf{n}}$-separable. So, by Anonymity, any subset of component of size n_s is $\tilde{\succsim}_{\mathbf{n}}$-separable. Also, by Pareto ($\succsim$ is regular), there must be at least one component in each $E^S(\mathbf{n})$, which is $\tilde{\succsim}_{\mathbf{n}}$-essential. By Anonymity, any component must be $\tilde{\succsim}_{\mathbf{n}}$-essential. By definition, this implies that any subset of components is $\tilde{\succsim}_{\mathbf{n}}$-essential.

Let us proof that any subset of components of size 2 is $\tilde{\succsim}_{\mathbf{n}}$-separable. There are three case:

Case 1: There exists $s \in S$ such that $n_s = 2$. Then the proof immediate.

Case 2: There exists $s, t \in S$ such that $n_s = n_t = 1$. By species separability, the subset $\{\sum_{k=1}^{s-1} n_k + 1, \sum_{k=1}^{t-1} n_k + 1\}$ is $\tilde{\succsim}_{\mathbf{n}}$-separable. So, by Anonymity, any subset of components of size 2 is $\tilde{\succsim}_{\mathbf{n}}$-separable.

Case 3: In other cases, there must exist s, $t \in S$ such that $n_s > 2$ and $n_t > 2$. The subset $\{\sum_{k=1}^{s-1} n_k + 1, \cdots, \sum_{k=1}^{s} n_k - 1, \sum_{k=1}^{t-1} n_k + 1\}$ has size n_s and is $\tilde{\succsim}_{\mathbf{n}}$-separable. The

[2] We use the conventtion that $\sum_{t=1}^{0} n_s = 0$.

subset $\{\sum_{k=1}^{t-1} n_k + 1, \cdots, \sum_{k=1}^{t} n_k - 1, \sum_{k=1}^{s-1} n_k + 1\}$ has size n_t and is $\widetilde{\succsim}_{\mathbf{n}}$-separable. The interaction of those two sets is $\{\sum_{k=1}^{t-1} n_k + 1, \sum_{k=1}^{s-1} n_k + 1\}$ and has size 2. By Lemma 2, $\{\sum_{k=1}^{t-1} n_k + 1, \sum_{k=1}^{s-1} n_k + 1\}$ is $\widetilde{\succsim}_{\mathbf{n}}$-separable. Given that this set has size 2, Anonymity implies that subset of components of size 2 is $\widetilde{\succsim}_{\mathbf{n}}$-separable.

By unions of subset of components of size 2 and with a non-empty intersection, we can obtain any subset of components. By Lemma 2, then, any subset of components is $\widetilde{\succsim}_{\mathbf{n}}$-separable. Therefore $\widetilde{\succsim}_{\mathbf{n}}$ is totally separable and continuous. Then, by Lemma 1, there exists there exists continuous functions $\phi_e^{\mathbf{n}} : X_e \to \mathbb{R}$ such that

$$\mathbf{x} \widetilde{\succsim}_{\mathbf{n}} \mathbf{y} \iff \sum_{e \in E} \phi_e^{\mathbf{n}}(x_e) \geq \sum_{e \in E} \phi_e^{\mathbf{n}}(y_e).$$

By Pareto, each $\phi_e^{\mathbf{n}}$ must be increasing, and by Anonymity they must be the same function $\tilde{\phi}_{\mathbf{n}}$. We can normalize the function so that $\tilde{\phi}_{\mathbf{n}}(0) = 0$.

Using the definition of $\widetilde{\succsim}_{\mathbf{n}}$, we obtain that for any a, $a' \in A_{\mathbf{n}}$:

$$a \succsim a' \iff \sum_{s=1}^{T} \sum_{i=1}^{n_s} \tilde{\phi}_{\mathbf{n}}(a_i(s)) \geq \sum_{s=1}^{T} \sum_{i=1}^{n_s} \tilde{\phi}_{\mathbf{n}}(a'_i(s)).$$

But we also know that

$$a \succsim a' \iff \sum_{s=1}^{T} V^s\Big(n_s, \Xi_{n_s}^s\big(a(s)\big)\Big) \geq \sum_{s=1}^{T} V^s\Big(n_s, \Xi_{n_s}^s\big(a'(s)\big)\Big).$$

By the unicity of additive representations up to a positive affine transformation (Debreu, 1960), there must exist $\chi_{\mathbf{n}}^1 \in \mathbb{R}_{++}$ and $\chi_{\mathbf{n}}^2 \in \mathbb{R}$ such that, for each $s \in S$ and $a \in A_{\mathbf{n}}$

$$V^s\Big(n_s, \Xi_{n_s}^s\big(a(s)\big)\Big) = \chi_{\mathbf{n}}^1 \times \left(\sum_{i=1}^{n_s} \tilde{\phi}_{\mathbf{n}}(a'_i(s)) \right) + \chi_{\mathbf{n}}^2.$$

Given that $V^s(n_s, 0) = 0$ and $\tilde{\phi}_{\mathbf{n}}(0) = 0$, we must have $\chi_{\mathbf{n}}^2 = 0$, so that, denoting $\phi_{\mathbf{n}} = \chi_{\mathbf{n}}^1 \tilde{\phi}_{\mathbf{n}}$:

$$V^s\Big(n_s, \Xi_{n_s}^s\big(a(s)\big)\Big) = \sum_{i=1}^{n_s} \phi_{\mathbf{n}}(a'_i(s)).$$

Gathering all our results so far, we obtain that there exist continuous increasing function $\phi_{\mathbf{n}} : \mathbb{R} \to \mathbb{R}$ such that $\phi_{\mathbf{n}}(0) = 0$ and for all $a, a' \in A$

$$a \succsim a' \iff \sum_{s=1}^{T} \sum_{i=1}^{n_s} \phi_{\mathbf{n}(a)}(a_i(s)) \geq \sum_{s=1}^{T} \sum_{i=1}^{n_s} \phi_{\mathbf{n}(a')}(a'_i(s)).$$

Consider any $x \in \mathbb{R}$ and $\mathbf{n} \in N^T$. Let a, $\hat{a} \in A$ be such that $a(1) = x$, $\hat{a}(1) = 0$ and $\hat{a}(2) = (x, \underbrace{0, \cdots, 0}_{n_2 - 1 \text{ times}})$. By Step 3 of the proof of Proposition 1 and species separability, $a_1 a^0_{\mathbf{n^1}} \sim a_1 a^0_{\mathbf{n}}$. By Anonymity, $a_1 a^0_{\mathbf{n}} \sim \hat{a}_{\{1,2\}} a^0_{\mathbf{n}}$. But again, by Step 3 of the proof of Proposition 1 and species separability, $\hat{a}_{\{1,2\}} a^0_{\mathbf{n}} \sim \hat{a}_2 a^0_{\mathbf{n}}$. So, by transitivity, $a_1 a^0_{\mathbf{n^1}} \sim \hat{a}_2 a^0_{\mathbf{n}}$. By our representation above, this means that $\phi(x) = \phi_{\mathbf{n}}(x)$. This is true for any $x \in \mathbb{R}$ and $\mathbf{n} \in N^T$.

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